

SINGULAR RATIONALLY CONNECTED SURFACES WITH NON-ZERO PLURI-FORMS

WENHAO OU

ABSTRACT. This paper is concerned with projective rationally connected surfaces X with canonical singularities and having non-zero pluri-forms, *i.e.* $H^0(X, (\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, where $(\Omega_X^1)^{[\otimes m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$. For such a surface, we can find a fibration from X to \mathbb{P}^1 . In addition, there is a surface Y with canonical singularities and a finite subgroup G of $\text{Aut}(Y)$ whose action is étale in codimension 1 over Y such that $X = Y/G$. Moreover there is a G -invariant fibration from Y to a smooth curve E with positive genus such that $\mathbb{P}^1 = E/G$ and $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(Y, (\Omega_Y^1)^{[\otimes m]})^G \cong H^0(E, (\Omega_E^1)^{\otimes m})^G$.

CONTENTS

1. Introduction and notation	1
2. Vanishing theorem for Fano varieties with Picard number 1	3
3. Mori fiber surfaces over a curve	4
3.1. Some properties of fibers	4
3.2. Singularities on non-reduced fibers	5
4. Proof of Theorem 1.3	7
4.1. Source of non-zero reflexive pluri-forms	8
4.2. Back to the initial variety	9
5. Proof of Theorem 1.2	9
6. Proof of Theorem 1.4	11
References	12

1. INTRODUCTION AND NOTATION

Recall that a projective variety X is said to be rationally connected if for any two general points in X , there exists a rational curve passing through them, see [Kol96, Def. 3.2 and Prop. 3.6]. It is known that for a smooth projective rationally connected variety X , $H^0(X, (\Omega_X^1)^{\otimes m}) = \{0\}$ for $m > 0$, see [Kol96, Cor. IV.3.8]. In [GKKP11, Thm. 5.1], it's shown that if a pair (X, D) is klt and X is rationally connected, then $H^0(X, \Omega_X^{[m]}) = \{0\}$ for $m > 0$, where $\Omega_X^{[m]}$ is the reflexive hull of Ω_X^m . On the other hand, by [GKP12, Thm. 3.3], if X is factorial, rationally connected and with canonical singularities, then $H^0(X, (\Omega_X^1)^{[\otimes m]}) = \{0\}$ for $m > 0$, where $(\Omega_X^1)^{[\otimes m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$. However, this is not true without the assumption of being factorial, see [GKP12, Example 3.7]. In this paper, our aim is to classify rationally connected surfaces with canonical singularities which have non-zero reflexive pluri-forms.

The following example is the one given in [GKP12, Example 3.7].

Example 1.1. Let $\pi' : X' \rightarrow \mathbb{P}^1$ be any smooth ruled surface. Choose four distinct points q_1, q_2, q_3, q_4 in \mathbb{P}^1 . For each point q_i , perform the following sequence of birational transformations of the ruled surface:

- (i) Blow up a point x_i in the fiber over q_i . Then we get two (-1) -curves which meet transversely at x'_i .
- (ii) Blow up the point x'_i . Over q_i , we get two disjoint (-2) -curves and one (-1) -curve. The (-1) -curve appears in the fiber with multiplicity two.
- (iii) Blow down the two (-2) -curves. We get two singular points on the fiber, each of them is of type A_1 .

In the end, we get a rationally connected surface $\pi : X \rightarrow \mathbb{P}^1$ with canonical singularities such that $H^0(X, (\Omega_X^1)^{[\otimes 2]}) \neq \{0\}$.

In fact, we will prove that every projective rationally connected surface X with canonical singularities and having non-zero pluri-forms can be constructed by a similar method from a smooth ruled surface over \mathbb{P}^1 . We have several steps:

- (i) Take a smooth ruled surface $\pi_0 : X_0 \rightarrow \mathbb{P}^1$ and choose distinct points q_1, \dots, q_r in \mathbb{P}^1 with $r \geq 4$.
- (ii) For each q_i , perform the same sequence of birational transformations as in Example 1.1. We get a fiber surface $\pi_1 : X_1 \rightarrow \mathbb{P}^1$. The non-reduced fibers of π_1 are $\pi_1^* q_1, \dots, \pi_1^* q_r$.
- (iii) Perform finitely many times this birational transformation: blow up a smooth point on a non-reduced fiber and then blow down the strict transform of the initial fiber. We obtain another fiber surface $p : X_f \rightarrow \mathbb{P}^1$.
- (iv) Starting from X_f , perform a sequence of blow-ups of smooth points, we get a surface X_a .
- (v) Blow down some chains of exceptional (-2) -curves for $X_a \rightarrow X_f$, the end product is our surface X .

We will prove the following result.

Theorem 1.2. *If X is a projective rationally connected surface with canonical singularities such that $H^0(X, (\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, then X can be constructed by the method described above.*

Note that we produce some non-reduced fibers over \mathbb{P}^1 during the process above. In fact, they are the source of non-zero forms.

Theorem 1.3. *Let X be a projective rationally connected surface with canonical singularities and having non-zero reflexive pluri-forms. If X_f be the result of a MMP, then X_f is a Mori fiber space over \mathbb{P}^1 . Let $p : X_f \rightarrow \mathbb{P}^1$ be the fibration. If r is the number of points over which p has non-reduced fibers, then we have $r \geq 4$ and*

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + \lfloor \frac{m}{2} \rfloor r))$$

for $m > 0$.

We note that both in Theorem 1.3 and in the construction of Theorem 1.2, we meet a surface named X_f . We will see later (in Proposition 5.3) that, by choosing a good MMP, these two surfaces are identical. On the other hand, there is a projective surface Y and a $4 : 1$ cover $\Gamma : Y \rightarrow X$. More precisely, we have the theorem below.

Theorem 1.4. *Let X be a projective rationally connected surface with canonical singularities and having non-zero pluri-forms. Let $X \rightarrow \mathbb{P}^1$ be the fibration given by Theorem 1.3. There is a smooth curve E with positive genus, a normal projective surface Y with canonical singularities such that Y is a fiber surface over E , $X \cong Y/G$, $\mathbb{P}^1 \cong E/G$, and $H^0(X, (\Omega_X^1)^{[\otimes m]})$ is isomorphic to the G -invariant part of $H^0(E, (\Omega_E^1)^{\otimes m})$, where $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ whose action is étale in codimension 1 over Y .*

Remark 1.5. With the notation in Theorem 1.4, Y is just the normalization of $X \times_{\mathbb{P}^1} E$ and we note that Y is not rationally connected. Moreover, if $r_Y : \tilde{Y} \rightarrow Y$ is the minimal resolution of Y , we will prove that $H^0(\tilde{Y}, (\Omega_{\tilde{Y}}^1)^{\otimes m}) \cong H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m})$, and the G -invariant part is isomorphic to $H^0(X, (\Omega_X^1)^{[\otimes m]})$.

$$\begin{array}{ccc}
 \tilde{Y} & & \\
 r_Y \downarrow & & \\
 Y & \xrightarrow[\text{4:1cover}]{\Gamma} & X \\
 \pi' \downarrow & & \downarrow \pi \\
 E & \xrightarrow[\text{4:1cover}]{\gamma} & \mathbb{P}^1
 \end{array}$$

Remark 1.6. Conversely, given a surface Y with canonical singularities, a finite subgroup G of $\text{Aut}(Y)$ whose action is étale in codimension 1 and a G -invariant fibration from Y to a smooth curve E with positive genus such that $E/G = \mathbb{P}^1$ such that every fiber is a chain of rational curves, we will have a rationally connected surface $X = Y/G$ such that $H^0(X, (\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$.

Throughout this paper, we will work over \mathbb{C} , the field of complex numbers. Unless otherwise specified, every variety is an integral \mathbb{C} -scheme of finite type. A curve is a variety of dimension 1 and a surface is a

variety of dimension 2. For a variety X , we denote the sheaf of Kähler differentials by Ω_X^1 . Denote $\bigwedge^p \Omega_X^1$ by Ω_X^p for $p \in \mathbb{N}$.

For a coherent sheaf \mathcal{F} on a variety X , we denote by \mathcal{F}^{**} the reflexive hull of \mathcal{F} . There is an important property for reflexive sheaves.

Proposition 1.7 ([Har80, Prop. 1.6]). *Let \mathcal{F} be a coherent sheaf on a normal variety V . Then \mathcal{F} is reflexive if and only if \mathcal{F} is torsion-free and for each open $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension at least 2, $\mathcal{F}(U) \cong j_* \mathcal{F}(U \setminus Y)$, where $j : U \setminus Y \rightarrow U$ is the inclusion map.*

If V is a normal variety, let V_{ns} be its smooth locus. We denote a canonical divisor by K_V . Moreover, let $\Omega_V^{[p]}$ (resp. $(\Omega_V^1)^{[\otimes p]}$) be the reflexive hull of Ω_V^p (resp. $(\Omega_V^1)^{\otimes p}$). By Proposition 1.7, it's the push-forward of the locally free sheaf $\Omega_{V_{ns}}^p$ (resp. $(\Omega_{V_{ns}}^1)^{\otimes p}$) to V since V is smooth in codimension 1.

Let S be a normal surface. Recall that a morphism $r : \tilde{S} \rightarrow S$ is called the minimal resolution of singularities (or minimal resolution for short) if \tilde{S} is smooth and $K_{\tilde{S}}$ is r -nef. There is a unique minimal resolution of singularities for a normal surface and any resolution of singularities factors through the minimal resolution.

Definition 1.8. Let S be a normal surface and let $r : \tilde{S} \rightarrow S$ be the minimal resolution of singularities of S . We say that S has *canonical singularities* if the intersection number $K_{\tilde{S}} \cdot C$ is zero for every r -exceptional curve C .

Remark 1.9. In [KM98, Def. 4.4], Definition 1.8 is the definition for *Du Val singularities*. However, by [KM98, Prop. 4.11 and Prop. 4.20], S has only Du Val singularities if and only if it has canonical singularities and it's automatically \mathbb{Q} -factorial. Thus these two definitions coincide. In this case, K_S is a Cartier divisor and $K_{\tilde{S}} = r^* K_S$. We know all Du Val singularities, they are A_i , D_j , E_k where $i \geq 1$, $j \geq 3$ and $k = 6, 7, 8$. For more details on Du Val singularities, see [KM98, §4.1].

Definition 1.10. Let $p : S \rightarrow B$ be a fibration from a normal surface to a smooth curve. If the non-reduced fibers of p are $p^* z_1, \dots, p^* z_r$, then the *ramification divisor* R of p is the divisor $p^* z_1 + \dots + p^* z_r - \text{Supp}(p^* z_1 + \dots + p^* z_r)$.

Let S be a projective rationally connected surface with canonical singularities, then we can run a minimal model program for S (for more details on MMP, see [KM98, §1.4 and §3.7]). We obtain a sequence of extremal contractions

$$S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_n.$$

Since K_S is not pseudo-effective, neither is K_{S_n} . Thus S_n is a Mori fiber space. we have a Mori fibration $p : S_n \rightarrow B$. Therefore we have two possibilities: either $\dim B = 0$ or $\dim B = 1$. If $\dim B = 0$, then S_n is a Fano variety with Picard number 1. Here, a Fano variety S is a normal projective variety such that $-K_S$ is an ample \mathbb{Q} -Cartier divisor. In §2, we will prove that S do not have any non-zero pluri-form in this case. Hence we will concern the case where $\dim B = 1$. In §3, we will study some properties for Mori fiber surfaces over a curve. In the end, we will prove Theorem 1.3, 1.2 and 1.4 successively in the last three sections.

2. VANISHING THEOREM FOR FANO VARIETIES WITH PICARD NUMBER 1

The aim of this section is to prove the theorem below.

Theorem 2.1. *Let V be a \mathbb{Q} -factorial klt Fano variety with Picard number 1, then $H^0(V, (\Omega_V^1)^{[\otimes m]}) = \{0\}$ for any $m > 0$.*

Before proving the theorem, we recall the notion of slopes. Let V be a normal projective \mathbb{Q} -factorial variety of dimension d . Let A be an ample divisor in V . Then for a coherent sheaf \mathcal{F} , we can define $\mu_A(\mathcal{F})$ the slope of \mathcal{F} with respect to A by

$$\mu_A(\mathcal{F}) := \frac{\det(\mathcal{F}) \cdot A^{d-1}}{\text{rank}(\mathcal{F})},$$

where $\det(\mathcal{F})$ is the reflexive hull of $\bigwedge^{\text{rank} \mathcal{F}} \mathcal{F}$. Moreover, let

$$\mu_A^{\max}(\mathcal{F}) = \sup\{\mu_A(\mathcal{G}) \mid \mathcal{G} \subseteq \mathcal{F} \text{ a coherent subsheaf}\}.$$

For any coherent sheaf \mathcal{F} , there is a saturated coherent subsheaf $\mathcal{G} \subseteq \mathcal{F}$ such that $\mu_A^{\max}(\mathcal{F}) = \mu_A(\mathcal{G})$, see [MP97, Prop. III.2.4].

Proposition 2.2. *Let V be a projective normal variety which is \mathbb{Q} -factorial, let H be an ample divisor in V , then for any two coherent sheaves \mathcal{F} and \mathcal{G} on V ,*

$$\mu_H^{\max}((\mathcal{F} \otimes \mathcal{G})^{**}) = \mu_H^{\max}(\mathcal{F}) + \mu_H^{\max}(\mathcal{G}).$$

For a proof of this proposition, see for instance [GKP12, Prop. A.16]. Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We may assume that $\dim V > 1$. We will argue by contradiction. Assume that there is a positive integer m such that $H^0(V, (\Omega_V^1)^{[\otimes m]}) \neq \{0\}$. Let H be an ample divisor on S .

Since $H^0(S, (\Omega_V^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, we have an injective morphism of sheaves from \mathcal{O}_S to $(\Omega_V^1)^{[\otimes m]}$. Thus $\mu_H^{\max}((\Omega_V^1)^{[\otimes m]}) \geq \mu_H^{\max}(\mathcal{O}_S) = 0$. Furthermore, by Proposition 2.2, we have $\mu_H^{\max}(\Omega_V^{[1]}) = m^{-1} \mu_H^{\max}((\Omega_V^1)^{[\otimes m]}) \geq 0$.

Therefore, there is a non-zero saturated coherent sheaf $\mathcal{F} \subseteq \Omega_V^{[1]}$ such that $\mu_H(\mathcal{F}) \geq 0$. Observe that $\text{rank}(\mathcal{F}) < \dim V$, otherwise $\mathcal{F} = \Omega_V^{[1]}$ and $\det(\mathcal{F}) \cong K_V$. Thus $\mu_H(\mathcal{F}) < 0$ a contradiction. We have two possibilities, either $\mu_H(\mathcal{F}) > 0$ or $\mu_H(\mathcal{F}) = 0$.

Case 1. Assume that $\mu_H(\mathcal{F}) > 0$. Since V has Picard number 1, $\det(\mathcal{F})$ is ample and its Kodaira-Iitaka dimension is $\dim V$. However this contradicts Bogomolov-Sommese vanishing theorem (see [GKKP11, Thm. 7.2]).

Case 2. Assume that $\mu_H(\mathcal{F}) = 0$. If $\mathcal{G} = \det \mathcal{F}$, then $\mathcal{G} \cdot H^{(\dim V - 1)} = 0$. Since V is \mathbb{Q} -factorial and klt, by [AD12, Lem. 2.6], there exists an integer l such that $(\mathcal{G}^{\otimes l})^{**}$ is isomorphic to \mathcal{O}_V . Let m be the smallest positive integer such that $(\mathcal{G}^{\otimes m})^{**} \cong \mathcal{O}_V$. We can construct the cyclic cover $q : Z \rightarrow V$ of V corresponding to \mathcal{G} , see [KM98, Def. 2.52]. Then $(q^* \mathcal{G})^{**} \cong \mathcal{O}_Z$. Since q is étale in codimension 1, Z is also klt by [Kol97, Prop. 3.16] and $-K_Z = q^*(-K_V)$ is ample. Thus Z is rationally connected by [HM07, Cor. 1.3 and 1.5]. And there are natural injective morphisms $(q^* \mathcal{G})^{**} \hookrightarrow (q^* \Omega_V^{[\text{rank}(\mathcal{F})]})^{**} \hookrightarrow \Omega_Z^{[\text{rank}(\mathcal{F})]}$. Hence we have an injection $\mathcal{O}_Z \hookrightarrow \Omega_Z^{[\text{rank}(\mathcal{F})]}$, but this contradicts [GKKP11, Thm. 5.1]. \square

3. MORI FIBER SURFACES OVER A CURVE

In this section, we study Mori fiber surfaces over a curve. In the first subsection, we will give some properties of the fibers. In the second subsection, we will classify the singularities on a non-reduced fiber.

We would like to introduce some notation for this section first. Let $p : S \rightarrow B$ be a Mori fibration, where B is a smooth curve and S is a normal surface with canonical singularities. Let $r : \tilde{S} \rightarrow S$ be the minimal resolution and $\tilde{p} = p \circ r : \tilde{S} \rightarrow B$.

Since S is singular at only finitely many points, p is smooth over general points of \mathbb{P}^1 and general fibers are all isomorphic to \mathbb{P}^1 . Note that a point in a smooth curve can also be regarded as a Cartier divisor and since any two fibers of p are numerically equivalent, we have $K_S \cdot p^*z = -2$ and $p^*z \cdot p^*z = 0$ for any $z \in \mathbb{P}^1$ by the adjunction formula.

3.1. Some properties of fibers.

Proposition 3.1. *If we run a \tilde{p} -relative MMP, we will get a smooth ruled surface over B in the end.*

Proof. Let $p_{S^m} : S^m \rightarrow B$ be the result of a \tilde{p} -relative MMP, then S^m is a smooth surface. Since $K_{\tilde{S}}$ is not pseudo-effective, neither is K_{S^m} . This implies that S^m is a ruled surface over B . \square

Proposition 3.2. *The support of p^*z is an irreducible Weil divisor for every $z \in B$.*

Proof. Assume the opposite and let C, D be 2 distinct components in p^*z which meet. Then $C \cdot D > 0$ and $C \cdot C < 0$ since $p^*z \cdot C = 0$. However, there is a positive number λ such that λC and D are numerically equivalent. That is $C \cdot C > 0$. Contradiction. \square

Proposition 3.3. *Let z be a point in B and let C be the support of p^*z . Then the coefficient of C in p^*z is at most equal to 2.*

Proof. Let $\alpha \in \mathbb{N}$ be the coefficient. Then $-2 = K_S \cdot p^*z = \alpha K_S \cdot C$. However, since K_S is a Cartier divisor, $K_S \cdot C \in \mathbb{Z}$. Thus $-2 \in \alpha \mathbb{Z}$ which means $\alpha \leq 2$. \square

Remark 3.4. If $p^*z = C$ as cycles, then C is smooth and S is smooth along C . First note that S is CM, since it is a normal surface. Then, in this case, $C = p^*z$ is a reduced subscheme since it is CM and generically reduced. By the adjunction formula (see [KM98, Prop. 5.73]) and the Riemann-Roch theorem (see [Har77, Ex. IV.1.9]), we have $2h^1(C, \mathcal{O}_C) - 2 = (K_S + p^*z) \cdot p^*z = -2$. This implies that $h^1(C, \mathcal{O}_C) = 0$ and C is isomorphic to \mathbb{P}^1 .

Proposition 3.5. *Let z be a point in B . If S is smooth along the support of p^*z , then p^*z is a reduced subscheme.*

Proof. Let C be the support of p^*z . Then C is an irreducible Cartier divisor by Proposition 3.2. By the adjunction formula, we have $2h^1(C, \mathcal{O}_C) - 2 = (K_S + C) \cdot C = K_S \cdot C < 0$.

Therefore, $K_S \cdot C = -2 = K_S \cdot p^*z$, which implies that $p^*z = C$ as cycles. Hence p^*z is a reduced subscheme in S for it is CM and generically reduced. \square

Proposition 3.6. *There exist at most 2 singular points of S on the fiber over $z \in B$.*

Proof. Let t be the number of singular points over $z \in B$ and assume that $t > 0$. Then the fiber p^*z is non-reduced. Let C be the support of p^*z and let \tilde{C} be its strict transform in \tilde{S} . Then $K_{\tilde{S}} \cdot \tilde{C} = 2^{-1}(K_{\tilde{S}} \cdot \tilde{p}^*z) = -1$, for $K_{\tilde{S}}$ is r -nef and \tilde{C} has coefficient 2 in \tilde{p}^*z . Let $E = \tilde{p}^*z - 2\tilde{C}$. Then $-1 = \tilde{C}^2 = 2^{-1}\tilde{C} \cdot (\tilde{p}^*z - E) = -2^{-1}\tilde{C} \cdot E$. Thus $t \leq \tilde{C} \cdot E = 2$. \square

3.2. Singularities on non-reduced fibers. In this subsection, we will give a list of non-reduced fibers of $p : S \rightarrow B$ (see Theorem 3.13). We will assume that p has non-reduced fiber over $0 \in B$, then there exist one or two singular points in S over $0 \in B$. We will study these two cases separately. We denote the support of p^*0 by C and its strict transform in \tilde{S} by \tilde{C} . First we will treat the case of two singular points.

Proposition 3.7. *Assume that there are two singular points over $0 \in B$, then each of them is of type A_1 .*

Proof. With the same notation as the proof of Proposition 3.6, we have $\tilde{C} \cdot E = 2$. Since there are 2 singular points over 0, we can decompose E into $D + D' + R$ such that $\tilde{C} \cdot D = \tilde{C} \cdot D' = 1$, $D \cdot D' = 0$ and $\tilde{C} \cdot R = 0$. Then $0 = \tilde{p}^*z \cdot D = 2\tilde{C} \cdot D + D^2 + D' \cdot D + R \cdot D = R \cdot D$. By symmetry, $R \cdot D' = 0$. Hence $R = 0$. \square

We will denote this type of fiber by $(A_1 + A_1)$. We note that this type of fiber does exist by Example 1.1. Next we will study the case of one singular point. In fact, we will prove that this isolated singularity is of type D_i ($i \geq 3$ and the type D_3 is just A_3). We would like to introduce some notation first.

Running a MMP relative to B for \tilde{S} gives a sequence:

$$\begin{array}{ccccccc} \tilde{S} & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_{n'} \\ & & & & & & \downarrow q_{n'} \\ & & & & & & B \end{array}$$

$\searrow \tilde{p}$

where $q_{n'} : Y_{n'} \rightarrow B$ is a Mori fibration by Proposition 3.1.

Recall the definition of dual graph. Let $C = \bigcup C_i$ be a collection of proper curves on a smooth surface S . The *dual graph* Γ of C is defined as follow:

- (1) The vertices of Γ are the curves C_i .
- (2) Two vertices $C_i \neq C_j$ are connected with $C_i \cdot C_j$ edges.

Proposition 3.8. *The support of \tilde{p}^*0 is a snc tree, i.e. it is a snc divisor and its dual graph is a tree.*

Proof. In fact, \tilde{S} can be obtained by a sequence of blow-ups from $Y_{n'}$. Thus the dual graph of the support of \tilde{p}^*0 is a snc tree. \square

Proposition 3.9. *The isolated singularity on the fiber over $0 \in B$ can only be of type D_i ($i \geq 3$).*

Proof. Let $C_0 = \tilde{C}$ and let $E_0 = \tilde{p}^*0 - 2C_0$. As in the proof of Proposition 3.6, we have $C_0^2 = -1$, $E_0^2 = -4$ and $E_0 \cdot C_0 = 2$. Since there is only one singular point over C and the support of \tilde{p}^*0 is a snc tree, we can decompose E_0 into $2C_1 + E_1$, where C_1 is a reduced irreducible rational curve. Since $2C_1 \cdot \tilde{p}^*0 = 0$

and $E_0^2 = -4$, we have $C_1^2 = -2$, $E_1^2 = -4$ and $C_1 \cdot E_1 = 2$. Thus the support of E_1 intersects C_1 at one or two points. If they intersect at two points, then as in Proposition 3.7, $E_1 = D + D'$ where D, D' are reduced irreducible rational curves, and we have $D \cdot D' = 0$, $D \cdot C_1 = 1$, $D' \cdot C_1 = 1$. If they intersect at one point, then we can decompose E_1 into $2C_2 + E_2$ where C_2 is reduced and irreducible, and we have $C_2^2 = -2$, $E_2^2 = -4$ and $E_2 \cdot C_2 = 2$. We are in the same situation as before. Hence by induction, we can decompose E_0 into $2(D_1 + \cdots + D_i) + D + D'$ where D, D' and all D_j are reduced and irreducible. Furthermore, we have $D \cdot D' = 0$, $D_i \cdot D = 1$, $D_i \cdot D' = 1$, $D_j \cdot D_{j+1} = 1$ for $1 \leq j \leq i-1$, and $D_j \cdot D_k = 0$ if $k - j > 1$. This shows that the singular point is of type D_{i+2} . \square

We denote these types of fibers by (D_i) ($i \geq 3$) according to their singularity. Now we will prove that these kinds of fibers exist.

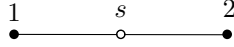
Proposition 3.10. *Let $x \in S$ be a smooth point over $0 \in B$ and let W be the blow-up of S at x with exceptional divisor $E \subseteq W$. Let D be the strict transform of C in W . Then we can blow down D and obtain another Mori fiber surface $S' \rightarrow B$.*

Proof. We have $C \cdot C = 0$, $K_S \cdot C = -1$, $K_W \cdot E = -1$, $E \cdot E = -1$ and $D \cdot E = 1$. Thus $K_W \cdot D = 0$ and $D \cdot D = -1$. Hence, by [KM98, Prop 4.10], we can blow down D and obtain another Mori fiber surface $S' \rightarrow B$. \square

We can use the operation in Proposition 3.10 to construct every type of non-reduced fibers.

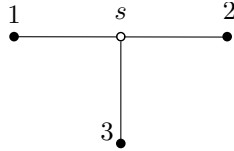
Proposition 3.11. *If S is of type $(A_1 + A_1)$ over $0 \in B$ then S' is of type (D_3) over $0 \in B$. If S is of type (D_i) over $0 \in B$ then S' is of type (D_{i+1}) over $0 \in B$ for $i \geq 3$.*

Proof. If the fiber is of type $(A_1 + A_1)$, the dual graph of the support of \tilde{p}^*0 in \tilde{S} is



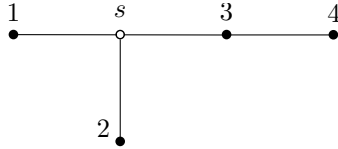
where s represents \tilde{C} .

Blow up a point only located in s , we get



Blow down successively 1, 2, s , we get S' . The special fiber is of type (D_3) .

On the other hand, instead of blowing down, if we continue to blow up at a point only located in 3, we get



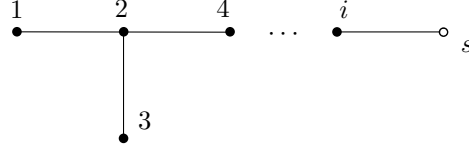
Blow down 1, 2, s , 3, we get a fiber of type (D_4) .

The same skill proves that if S has special fiber of type (D_i) then S' is of type (D_{i+1}) over $0 \in B$. \square

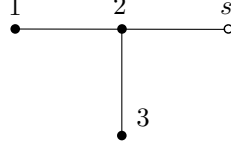
In fact, the fibers obtained in the proposition above are all possible non-reduced fibers with one singular point.

Proposition 3.12. *All non-reduced fibers in S can be obtained by the methods described in Example 1.1 and Proposition 3.11.*

Proof. To see this, it's enough to describe the dual graph of the support of $\tilde{p}^*0 \subseteq \tilde{S}$. If the fiber is of type (D_i) ($i \geq 4$) then this dual graph must be



where s represents \tilde{C} . We run a \tilde{p} -relative MMP around this fiber, then the first curve contracted must be s since other curves are all (-2) -curves. Now i becomes a (-1) -curve. If we contract the curves $1, \dots, i-1$, we get a fiber of type (D_{i-1}) . Thus by induction we may now assume that the fiber is of type (D_3) . The dual graph of the support of $\tilde{p}^*0 \subseteq \tilde{S}$ is as below.



In a MMP, we contract s at first, then the remaining fiber is just the same as the one in the second step of Example 1.1. This ends the proof. \square

In the end, we obtain a table of non-reduced fibers.

Theorem 3.13. *Let S be a quasi-projective surface with canonical singularities and let B be a smooth curve such that there is a Mori fibration $p : S \rightarrow B$ which has non-reduced fiber over $0 \in B$. Let $r : \tilde{S} \rightarrow S$ be the minimal resolution and \tilde{p} be $p \circ r$, then we have a table*

Type of fiber	Dual graph
$(A_1 + A_1)$	
(D_3)	
(D_i)	

where the dual graph is the one of the support of $\tilde{p}^*0 \subseteq \tilde{S}$ and s corresponds to \tilde{C} .

4. PROOF OF THEOREM 1.3

We will first prove Theorem 1.3. Let X be a rationally connected projective normal surface such that X has canonical singularities and $H^0(X, \Omega_X^{[\otimes m]}) \neq \{0\}$ for some $m > 0$. Run a MMP for X . We will get a sequence of divisorial contractions

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = X_f.$$

Let $X_{i,ns}$ be the smooth locus of X_i .

Proposition 4.1. *For $m \in \mathbb{N}$, there is an injection*

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \hookrightarrow H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]})$$

Proof. For any $i \in \{0, \dots, n\}$, X_i is normal, thus by Proposition 1.7,

$$H^0(X_i, (\Omega_{X_i}^1)^{[\otimes m]}) \cong H^0(X_{i,ns}, (\Omega_{X_{i,ns}}^1)^{\otimes m}).$$

Since $X_i \rightarrow X_{i+1}$ is a divisorial contraction, $X_{i+1,ns} \setminus \{x_{i+1}\}$ is isomorphic to an open subset of $X_{i,ns}$ where $\{x_{i+1}\} \subseteq X_{i+1}$ is the image of the exceptional divisor. This gives rise to an injection

$$H^0(X_{i,ns}, (\Omega_{X_{i,ns}}^1)^{\otimes m}) \hookrightarrow H^0(X_{i+1,ns} \setminus \{x_{i+1}\}, (\Omega_{X_{i+1,ns} \setminus \{x_{i+1}\}}^1)^{\otimes m}) \cong H^0(X_{i+1,ns}, (\Omega_{X_{i+1,ns}}^1)^{\otimes m})$$

by Proposition 1.7. Then we have $H^0(X_i, (\Omega_{X_i}^1)^{[\otimes m]}) \hookrightarrow H^0(X_{i+1}, (\Omega_{X_{i+1}}^1)^{[\otimes m]})$. The composition of these injections is just $H^0(X, (\Omega_X^1)^{[\otimes m]}) \hookrightarrow H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]})$. \square

Let $f : X \rightarrow X_f$ be the composition of the sequence of the MMP, then $H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \neq \{0\}$. Thus X_f is a Mori fiber surface over a normal rationally connected curve by Theorem 2.1. We have a fibration $p : X_f \rightarrow \mathbb{P}^1$. Let $\pi = p \circ f : X \rightarrow \mathbb{P}^1$.

4.1. Source of non-zero reflexive pluri-forms. In this subsection, we will find out the source of non-zero pluri-forms on X_f . By Proposition 1.7, we have $H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \cong H^0(U, (\Omega_U^1)^{\otimes m})$, where $m \in \mathbb{N}$ and U is any open subset of $X_{f,ns}$, the smooth locus of X_f , such that $X_f \setminus U$ has codimension at least 2 in X_f .

On the other hand, we have a natural morphism of locally free sheaves on $X_{f,ns}$, $(p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \rightarrow \Omega_{X_{f,ns}}^1$. Furthermore, if R is the ramification divisor of $p|_{X_{f,ns}}$ there exist a factorisation

$$(p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \rightarrow ((p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1) \otimes \mathcal{O}_{X_{f,ns}}(R) \xrightarrow{\rho} \Omega_{X_{f,ns}}^1.$$

Moreover, $\rho \otimes k_x$ is injective for x in an open subset $V \subseteq X_{f,ns}$ such that $X_{f,ns} \setminus V$ is a finite set of points, where k_x is the residue field of x . Thus we have an exact sequence

$$0 \rightarrow ((p|_V)^* \Omega_{\mathbb{P}^1}^1) \otimes \mathcal{O}_V(R|_V) \rightarrow \Omega_V^1 \rightarrow \mathcal{G} \rightarrow 0,$$

where $\mathcal{G} = \Omega_{V/\mathbb{P}^1}^1 / (\text{torsion of } \Omega_{V/\mathbb{P}^1}^1)$ is an invertible sheaf on V , for $\mathcal{G} \otimes k_x$ is of rank 1 at every point x of V , where k_x is the residue field of x . By [Har77, Ex. III.5.16], there is a filtration over V :

$$(\Omega_V^1)^{\otimes m} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots \supseteq \mathcal{F}_m \supseteq \mathcal{F}_{m+1} = 0,$$

such that $\mathcal{F}_i / \mathcal{F}_{i+1} \cong \mathcal{G}^{\otimes (m-i)} \otimes (((p|_V)^* \Omega_{\mathbb{P}^1}^1) \otimes \mathcal{O}_V(R|_V))^{\otimes i}$ for every $i \in \{0, \dots, m\}$.

Proposition 4.2. *With the notation above, we have a natural isomorphism*

$$H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m) \otimes (p|_{X_{f,ns}})^* \mathcal{O}_{X_{f,ns}}(mR)).$$

Proof. For $z \in \mathbb{P}^1$ a general point, the support C of the fiber p^*z is isomorphic to \mathbb{P}^1 . We have $\mathcal{G}|_C \cong \mathcal{O}_C(-2)$ and $((p|_V)^* \Omega_{\mathbb{P}^1}^1) \otimes \mathcal{O}_V(R|_V)|_C \cong \mathcal{O}_C$ for p is smooth along C . Thus $(\mathcal{F}_i / \mathcal{F}_{i+1})|_C \cong \mathcal{O}_C(2(i-m))$ for $i < m$. Hence $H^0(V, \mathcal{F}_i / \mathcal{F}_{i+1}) = 0$ and $H^0(V, \mathcal{F}_i) \cong H^0(V, \mathcal{F}_{i+1})$ for $i < m$.

On the other hand, by Proposition 1.7, we have

$$H^0(V, ((p|_V)^* \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_V(R|_V))^{\otimes m}) \cong H^0(X_{f,ns}, ((p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_{X_{f,ns}}(R))^{\otimes m}),$$

and

$$H^0(X_{f,ns}, ((p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_{X_{f,ns}}(R))^{\otimes m}) \cong H^0(\mathbb{P}^1, (p|_{X_{f,ns}})^* ((p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_{X_{f,ns}}(R))^{\otimes m})$$

which is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m) \otimes (p|_{X_{f,ns}})^* \mathcal{O}_{X_{f,ns}}(mR))$ by the projection formula. Finally we obtain

$$H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \cong H^0(V, \mathcal{F}_m) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m) \otimes (p|_{X_{f,ns}})^* \mathcal{O}_{X_{f,ns}}(mR)).$$

\square

We note that $(p|_{X_{f,ns}})^* \mathcal{O}_{X_{f,ns}}(mR)$ is a torsion-free sheaf of rank 1 on \mathbb{P}^1 , thus it's an invertible sheaf and there is a $k \in \mathbb{Z}$ such that $\mathcal{O}_{\mathbb{P}^1}(k)$ is isomorphic to $(p|_{X_{f,ns}})^* \mathcal{O}_{X_{f,ns}}(mR)$. If this k is not less than $2m$, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m) \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \neq \{0\}$ and there exist non-zero reflexive pluri-forms over X_f .

Proposition 4.3. *Assume that the non-reduced fibers of $p : X_f \rightarrow \mathbb{P}^1$ are over z_1, \dots, z_r . Then for $m \in \mathbb{N}$, we have $(p|_{X_{f,ns}})_* \mathcal{O}_{X_{f,ns}}(mR) \cong \mathcal{O}_{\mathbb{P}^1}(\lfloor \frac{m}{2} \rfloor (z_1 + \dots + z_r)) \cong \mathcal{O}_{\mathbb{P}^1}(\lfloor \frac{m}{2} \rfloor r)$, where $\lfloor \cdot \rfloor$ is the integer part. In particular, $H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + \lfloor \frac{m}{2} \rfloor r))$.*

Proof. Since every two points in \mathbb{P}^1 are linearly equivalent and every irreducible component of R is contained in a fiber, we may assume that $r = 1$ for simplicity. By Proposition 3.2 and Proposition 3.3, R is irreducible and $(p|_{X_{f,ns}})^* z_1 = 2R$. Assume that $(p|_{X_{f,ns}})_* \mathcal{O}_{X_{f,ns}}(mR) \cong \mathcal{O}_{\mathbb{P}^1}(k \cdot z_1)$, we have to prove that $k = \lfloor \frac{m}{2} \rfloor$.

We note that $\gamma \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k \cdot z_1))$ is just a rational function on \mathbb{P}^1 which can only have pole at z_1 with multiplicity at most k . Its pull-back to $X_{f,ns}$ is a rational function which can only have pole along R with multiplicity at most $2k$. Thus k is the largest integer such that $2k \leq m$, i.e. $k = \lfloor \frac{m}{2} \rfloor$. \square

4.2. Back to the initial variety. We have studied X_f and now we have to reverse the MMP and pull back reflexive pluri-forms to the initial variety X . In this subsection, we will prove that $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]})$ which ends the proof of Theorem 1.3.

Let $f : X \rightarrow X_f$ be the composition of the sequence in the MMP and let $\pi = p \circ f : X \rightarrow \mathbb{P}^1$. Assume that the non-reduced fibers of p are $p^* z_1, \dots, p^* z_r$ and the ones of π are $\pi^* z_1, \dots, \pi^* z_r, \pi^* z'_1, \dots, \pi^* z'_t$. We note that the fibers of $\pi : X \rightarrow \mathbb{P}^1$ have reduced components over $z'_1, \dots, z'_t \in \mathbb{P}^1$ since $p : X_f \rightarrow \mathbb{P}^1$ has reduced fibers over these points. Thus the ramification divisor over these points will not give contribution to non-zero reflexive pluri-forms. Our aim is now to prove that $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X_f, (\Omega_{X_f}^1)^{[\otimes m]})$. To achieve this, it's enough to prove that the fibers of $\pi : X \rightarrow \mathbb{P}^1$ over z_1, \dots, z_r are non-reduced along each of their components, i.e. the coefficient of any component in $\pi^*(z_1 + \dots + z_r)$ is larger than 1.

Proposition 4.4. *Let S be a projective surface which has at most canonical singularities. Let $c : S \rightarrow S_1$ be a extremal contraction which contracts a divisor E to a point x . Then S_1 is smooth at x .*

Proof. We suppose the opposite. Let $r_S : \tilde{S} \rightarrow S$ and $r_{S_1} : \tilde{S}_1 \rightarrow S_1$ be minimal resolutions, we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{c}} & \tilde{S}_1 \\ r_S \downarrow & & \downarrow r_{S_1} \\ S & \xrightarrow{c} & S_1 \end{array}$$

Let \tilde{E} be the strict transform of E in \tilde{S} . Then $K_{\tilde{S}} \cdot \tilde{E} = r^* K_S \cdot \tilde{E} = K_S \cdot r_{S*} \tilde{E} = K_S \cdot E < 0$. Thus \tilde{E} must be contracted by \tilde{c} . Since E is over x , $\tilde{c}(\tilde{E})$ is in an exceptional divisor D of $r_{S_1} : \tilde{S}_1 \rightarrow S_1$. Let \tilde{D} be the strict transform of D in \tilde{S} . Then \tilde{D} is contracted by r_S for $\tilde{D} \neq \tilde{E}$. Thus \tilde{D} is a (-2) -curve in \tilde{S} . On the other hand, since \tilde{E} is over a point of D , \tilde{c} will contract a divisor in \tilde{S} which meets \tilde{D} . Hence the image of \tilde{D} by \tilde{c} , D , has self-intersection number larger than (-2) . Contradiction. \square

From Proposition 4.4, every exceptional divisor of $f : X \rightarrow X_f$ is over a smooth point of X_f .

Proposition 4.5. *The fibers of $\pi : X \rightarrow \mathbb{P}^1$ over $z_1, \dots, z_r \in \mathbb{P}^1$ are non-reduced along each of their components.*

Proof. With the same notation as Proposition 4.4, we denote the set of singular points of X_f by $\{x_1, \dots, x_u\}$. Then $X_{f,ns} = X_f \setminus \{x_1, \dots, x_u\}$. Let $X_1 = f^{-1}(X_{f,ns}) \subseteq X$. Since f is an isomorphism around the points $f^{-1}(x_1), \dots, f^{-1}(x_u)$, the open subset $r^{-1}(X_1) \subseteq \tilde{X}$ can be obtain by a sequence of blow-ups of smooth points from $r_f^{-1}(X_{f,ns}) \cong X_{f,ns}$. Thus the fibers of $(\pi \circ r)|_{r^{-1}(X_1)}$ over z_1, \dots, z_r are non-reduced along each of their components. Hence so are the fibers of π over these points. Moreover, in each $\pi^* z_i$, there is a component having coefficient 2. \square

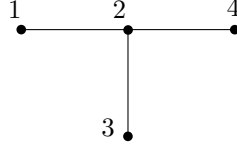
From Proposition 4.5, we obtain Theorem 1.3.

5. PROOF OF THEOREM 1.2

We will now prove Theorem 1.2. If X is a rationally connected projective surface such that X has canonical singularities and $H^0(X, (\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ and X_f is the result of a MMP, then X and X_f

are isomorphic around the singular locus of X_f by Proposition 4.4. The proof of Proposition 4.5 gives us an idea of how to reconstruct X from X_f . First we construct a surface X_a which can be obtained from X_f by a sequence of blow-ups of smooth points. Then we blow down some chains of exceptional (-2) -curves for $X_a \rightarrow X_f$ and we obtain X . To do this, we have to study the structure of exceptional divisors for $X_a \rightarrow X_f$.

Proposition 5.1. *Denote a germ of smooth surface by $(0 \in S)$. Let $h : S' \rightarrow S$ be the composition of a sequence of blow-ups of smooth points over $0 \in S$. Let D be the support of h^*0 . Then any (-2) -curve in D meets at most 2 other (-2) -curves. In another word, the dual graph of D cannot contain a subgraph as below such that each vertex of the subgraph corresponds a (-2) -curve.*



Proof. Assume the opposite. We know that we can reverse the process of blow-ups by running a MMP relatively to S . Thus these four curves will be successively contracted during the MMP. The first one contracted cannot be 2, since the remaining should make up a tree by an analogue result of Proposition 3.8. By symmetry, assume that 1 is first contracted. If 3 (or 4) is contracted secondly, the self-intersection number of 2 is at least 0. If 2 is contracted secondly, a further contraction will also produce a curve with self-intersection number at least 0. But this curve is over $0 \in S$, it must have negative self-intersection number by the negativity theorem (see [KM98, Lem. 3.40]). Contradiction. \square

In particular, every connected collection of (-2) -curves in D has a dual graph as below



Moreover if we contract such a chain we will produce a singular point of type A_i .

We will now prove that it's possible to contract such a chain.

Proposition 5.2. *Let $C = \bigcup_{1 \leq k \leq i} C_k$ be a chain of (-2) -curves as above in a smooth surface S . Then there exists a morphism $c : S \rightarrow S'$ such that S' has canonical singularities and c contracts exactly C .*

Proof. We note that $K_S \cdot C_k = 0$ for every k . Thus it is enough to prove that the intersection matrix $\{C_k \cdot C_j\}$ is negative definite by [KM98, Prop 4.10]. We have $C_k \cdot C_k = -2$, $C_k \cdot C_{k+1} = 1$ and $C_k \cdot C_{j+1} = 0$ if $k < j$. Hence it's enough to prove that for any $(x_1, \dots, x_i) \in \mathbb{R}^i \setminus \{0\}$,

$$2(x_1^2 + \dots + x_i^2) - 2(x_1x_2 + \dots + x_{i-1}x_i) > 0.$$

However, the left-hand side is just $x_1^2 + x_i^2 + (x_1 - x_2)^2 + \dots + (x_{i-1} - x_i)^2$, which is positive for $(x_1, \dots, x_i) \in \mathbb{R}^i \setminus \{0\}$. \square

Now we can conclude Theorem 1.2. For reconstructing a rationally connected surface X with canonical singularities and having non-zero pluri-forms, we will reverse the MMP. First of all, we take a ruled surface X_0 over \mathbb{P}^1 . By producing (at least 4) non-reduced fibers by the skill of Example 1.1 and Proposition 3.11, we obtain a canonical surface X'_f (which is isomorphic to X_f , the result of a MMP, as we will see below). Then, starting from X'_f , we blow up successively smooth points and we get a surface X_a . Finally, we contract chains of (-2) -curves which are exceptional for $X_a \rightarrow X'_f$. This is always possible by Proposition 5.2. We obtain X and we have a natural morphism $f' : X \rightarrow X'_f$ since every curve contracted in $X_a \rightarrow X$ is also contracted in $X_a \rightarrow X'_f$. It remains to prove

Proposition 5.3. *With the notation above, if we run a MMP for X , we can obtain X'_f in the end as a Mori firer space.*

Proof. We first run a f' -relative MMP for X , and we have $f_r : Y \rightarrow X'_f$ in the end. Let's proof that f_r is an isomorphism. Let $r_Y : \tilde{Y} \rightarrow Y$ be the minimal resolution. Since $K_{\tilde{Y}}$ is r_Y -nef and K_Y is f_r -nef, $K_{\tilde{Y}}$ is $(f_r \circ r_Y)$ -nef. Thus $\tilde{Y} \rightarrow X'_f$ is the minimal resolution of X'_f and we have $X'_f \cong Y$ since X'_f and Y are isomorphic around the singular locus of X'_f . \square

6. PROOF OF THEOREM 1.4

We would like to prove Theorem 1.4 in this section. In [GKP12, Remark and Question 3.8], for X in Example 1.1, we can find a smooth elliptic curve E , a smooth ruled surface Y (which is \tilde{X} in [GKP12]) such that \mathbb{P}^1 is the quotient of E by $\mathbb{Z}/2\mathbb{Z}$ and X is the quotient of Y by the same group. In this section, we would like to construct such a surface Y for any rationally connected surface X with canonical singularities and having non-zero pluri-forms.

We will first construct the curve E .

Proposition 6.1. *let q_1, \dots, q_r be r different points on \mathbb{P}^1 with $r \geq 4$, then there exist a smooth curve E , a $4 : 1$ cover $\gamma : E \rightarrow \mathbb{P}^1$ such that γ is exactly ramified at 2 points over each q_i , $i = 1, \dots, r$.*

Proof. Since $r \geq 4$, we can find an elliptic curve E' and a $2 : 1$ cover $\gamma' : E' \rightarrow \mathbb{P}^1$ such that γ' is ramified exactly over q_1, q_2, q_3, q_4 . If $r > 4$, assume that $\gamma'^{-1}(\{q_i\}) = \{q'_{i,1}, q'_{i,2}\}$ for $i > 4$ and $\gamma'^{-1}(\{q_1\}) = \{q'_1\}$. Then $\mathcal{O}_{E'}(2(r-4)q'_1)$ is isomorphic to $\mathcal{O}_{E'}(\sum_{i>4} q'_{i,1} + \sum_{i>4} q'_{i,2})$. Thus we can construct a ramified cyclic cover of E , with respect to the line bundle $\mathcal{O}_{E'}((r-4)q'_1)$, $\beta : E \rightarrow E'$, such that E is smooth, β is of degree 2 and ramified exactly over $q'_{i,1}, q'_{i,2}$, $i > 4$, see [KM98, Def. 2.50].

If $r = 4$, let $\gamma'^{-1}(\{q_2\}) = \{q'_2\}$. Then $\mathcal{O}_{E'}(2q'_1 - 2q'_2) \cong \mathcal{O}_{E'}$ and we can construct a cyclic cover of E , with respect to the non-trivial line bundle $\mathcal{O}_{E'}(q'_1 - q'_2)$, $\beta : E \rightarrow E'$. E is a smooth elliptic curve and β is étale of degree 2.

Finally, the composition $\gamma = \gamma' \circ \beta : E \rightarrow \mathbb{P}^1$ is a $4 : 1$ cover, and exactly ramified at 2 points over each $q_i \in \mathbb{P}^1$. Moreover \mathbb{P}^1 is the quotient E/G where $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. \square

The surface Y we want is in fact the normalization of the pull-back of X by the base extension $\gamma : E \rightarrow \mathbb{P}^1$. First let Y_f be the normalization of the pull-back of X_f by the base extension $\gamma : E \rightarrow \mathbb{P}^1$. Then the induced fibration $\pi'_f : Y_f \rightarrow E$ has reduced fibers. To see this, we may calculate with local coordinates. For example, assume that $\pi_f : X_f \rightarrow \mathbb{P}^1$ has non-reduced fiber over q_1 and let x be a smooth point of X_f on this fiber. Let a be a local coordinate of \mathbb{P}^1 , (a_X, b_X) be local coordinates of X_f and a_E be a local coordinate of E such that π_f corresponds to $(a_X, b_X) \mapsto a_X^2$ and γ corresponds to $a_E \mapsto a_E^2$. Then the pull-back of X by the base extension $\gamma : E \rightarrow \mathbb{P}^1$ is locally the same as $\text{Spec}(\mathbb{C}[a_E, a_X, b_X]/(a_E^2 - a_X^2))$. The normalisation Y_f is locally the same as $\text{Spec}(\mathbb{C}[a_E, b_X, c]/(c^2 - 1))$, where c is just $a_E \cdot a_X^{-1}$. And $\pi'_f : Y_f \rightarrow E$ corresponds locally to $(a_E, b_X, c) \mapsto a_E$. Thus $\pi'_f : Y_f \rightarrow E$ has reduced fibers, moreover, we see that $\Gamma_f : Y_f \rightarrow X_f$ is étale over the smooth locus of X_f .

Now we reconstruct X from X_f . Since $\Gamma_f : Y_f \rightarrow X_f$ is étale over the smooth locus of X_f , every operation we do with X_f can be done in the analogue way with Y_f . Thus the surface Y we obtained in this way is the one we want in Theorem 1.4 and it is the normalization of the pull-back of X by the base extension $\gamma : E \rightarrow \mathbb{P}^1$. There is an action of G on Y induced by the one of G on E and we have $X = Y/G$.

We note that the general fibers of π' are smooth rational curves in Y , thus for general $z \in E$, we have $(\Omega_{Y/E}^1)|_{\pi'^{-1}(\{z\})} \cong \mathcal{O}_{\pi'^{-1}(\{z\})}(-2)$. Moreover, every fiber of π' is reduced along one of its components since $Y_f \rightarrow E$ has reduced fibers, therefore by the same argument as Proposition 4.2, $H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m})$. On the other hand, we have

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]})^G \cong H^0(X, (\Omega_X^1)^{[\otimes m]}),$$

thus $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m})^G$. We want to show the right hand side of the last equality is just $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}](q_1 + \dots + q_r)))$. To achieve this, we need the proposition below

Proposition 6.2. *Let R_γ be the ramification divisor of $\gamma : E \rightarrow \mathbb{P}^1$, then $(\gamma_* \mathcal{O}_E(R_\gamma))^G \cong \mathcal{O}_{\mathbb{P}^1}$.*

Proof. We have $H^0(U, (\gamma_* \mathcal{O}_E(R_\gamma))^G) \cong H^0(\gamma^{-1}U, \mathcal{O}_E(R_\gamma))^G$ for any open set $U \subseteq \mathbb{P}^1$. Let θ be a rational function on E such that θ represents a non-zero element in $H^0(\gamma^{-1}U, \mathcal{O}_E(R_\gamma))^G$. Since θ is G -invariant, it can also be regarded as a rational function on U . Since θ can only have simple poles at the support of R on $\gamma^{-1}U$, it cannot have any pole on U . Thus $(\gamma_* \mathcal{O}_E(R_\gamma))^G \cong \mathcal{O}_{\mathbb{P}^1}$. \square

We have $(\Omega_E^1)^{\otimes m} \cong (\gamma^* \mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}](q_1 + \dots + q_r))) \otimes \mathcal{O}_E((m - 2[\frac{m}{2}])R_\gamma)$. Thus

$$H^0(E, (\Omega_E^1)^{[\otimes m]})^G \cong H^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}r]) \otimes (\gamma_* \mathcal{O}_E((m - 2[\frac{m}{2}])R_\gamma))^G).$$

By the previous proposition, we obtain $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$, which is the same as Proposition 4.3.

Remark 6.3. If $r_Y : \tilde{Y} \rightarrow Y$ be minimal resolution of Y , then we have $H^0(\tilde{Y}, (\Omega_{\tilde{Y}}^1)^{\otimes m}) \cong H^0(E, (\Omega_E^1)^{[\otimes m]}) \cong H^0(Y, (\Omega_Y^1)^{[\otimes m]})$ by the same argument as Proposition 4.2. We obtain Theorem 1.4 and we have a commutative diagram as below

$$\begin{array}{ccc}
 \tilde{Y} & & \\
 r_Y \downarrow & & \\
 Y & \xrightarrow[\text{4:1cover}]{\Gamma} & X \\
 \pi' \downarrow & & \downarrow \pi \\
 E & \xrightarrow[\text{4:1cover}]{\gamma} & \mathbb{P}^1
 \end{array}$$

Example 6.4. In the end, we will give some examples. Let $h(m, r)$ be the dimension of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$. Then

If $r = 4$, then $h(m, 4) = 1$ if $m > 0$ is even and $h(m, 4) = 0$ if m is odd.

If $r = 5$, then $h(2, 5) = 2$, $h(3, 5) = 0$ and $h(m, 5) > 0$ if $m \geq 4$.

If $r \geq 6$, then $h(m, r) > 0$ for $m \geq 2$.

REFERENCES

- [AD12] Carolina Araujo and Stéphane Druel, *On codimension 1 del pezzo foliations on varieties with mild singularities*, Preprint [arXiv:1210.4013](#), 2012.
- [GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 87–169. MR 2854859
- [GKP12] Daniel Greb, Stefan Kebekus, and Thomas Peternell, *Reflexive differential forms on singular spaces – geometry and cohomology*, Preprint [arXiv:1202.3243v2](#), 2012.
- [Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
- [Har80] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176. MR 597077 (82b:14011)
- [HM07] Christopher D. Hacon and James Mckernan, *On Shokurov’s rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136. MR 2309156 (2008f:14030)
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)
- [Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)
- [Kol97] ———, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, AMS, 1997, pp. 221–287.
- [MP97] Yoichi Miyaoka and Thomas Peternell, *Geometry of higher-dimensional algebraic varieties*, DMV Seminar, vol. 26, Birkhäuser Verlag, Basel, 1997. MR 1468476 (98g:14001)

WENHAO OU: INSTITUT FOURIER, UMR 5582 DU CNRS, UNIVERSITÉ GRENoble 1, BP 74, 38402 SAINT MARTIN D’HÈRES, FRANCE